Velocity of Ultrasonic Waves in Water by the Debye-Sears Effect and the Bulk Modulus of Water

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October 31, 2023

Abstract

In this experiment, the order of diffractions produced by a sodium lamp was studied to determine the wavelength of the sodium lamp, λ_L, the velocity of sound, v_s, and the bulk modulus of water, B. This was achieved by using a standard diffraction grating and an ultrasonic standing wave in water as an optical diffraction grating and measuring the difference in angle, $\Delta\theta$, between diffraction orders. Through this, λ_L was found to be 606.08 ± 0.00747nm, v_s was found to be $1.905.3 \pm 45.6\frac{\text{m}}{\text{s}}$, which agrees within 21.4% of the literature value of $1497.5 \pm 0.5\frac{\text{m}}{\text{s}}$ [3], and B was found to be 3.63 ± 0.17GPa, which agrees within 41.7% of the literature value of 1.96 GPa [4]. Hence, the experimental method was applied somewhat successfully to determine the values of v_s and B.

1 Introduction [1]

Acoustic waves propagating within a liquid medium can cause intrinsic properties like density to change with respect to the spacing due to pressure nodes and antinodes. This can be determined by the velocity (v_s) and frequency of the sound wave (f_s). The purpose of this lab is to experimentally determine the velocity of ultrasonic waves in water and the bulk modulus of water for adiabatic compression. This is achieved by applying the Deybe-Sears Effect and the Bulk Modulus Equation.

1.1 The Debye-Sears Effect

For ultrasonic waves, especially those with frequencies in the MHz range, the spacing between low density and high density regions within a liquid medium like water are comparable to the spacing used in diffraction gratings. As the index of refraction of liquid is dependent on its density, variations in density across the spacing will yield variations in the refractive index, which is evident when a monochromatic parallel light beam passing perpendicularly to the sound wave will refract as if it were passing through a diffraction grating with spacing d such that $d = \lambda_s$.

The relationship between the order of diffraction (m), the spacing of the grating (d), the diffraction angle (θ) and the wavelength of the source (λ) is given by:

$$m\lambda = dsin(\theta) \tag{1}$$

The velocity of the ultrasonic wave can be modelled by the relationship:

$$v_s = \lambda_s f_s \tag{2}$$

1.2 The Bulk Modulus Equation

The speed of sound (v_s) is dependent on two properties of the medium: the inertial property (which describes the kinetic energy storage) and the elastic property (which describes the potential energy storage). The bulk modulus equation relates the speed of sound to its inertial and elastic properties given by the expression below:

$$v_s = \sqrt{\frac{B}{\rho_{water}}} \tag{3}$$

where: v_s = speed of ultrasonic sound waves in water, B = bulk modulus of water and ρ = density of water.

2 Materials and Methods

2.1 Materials

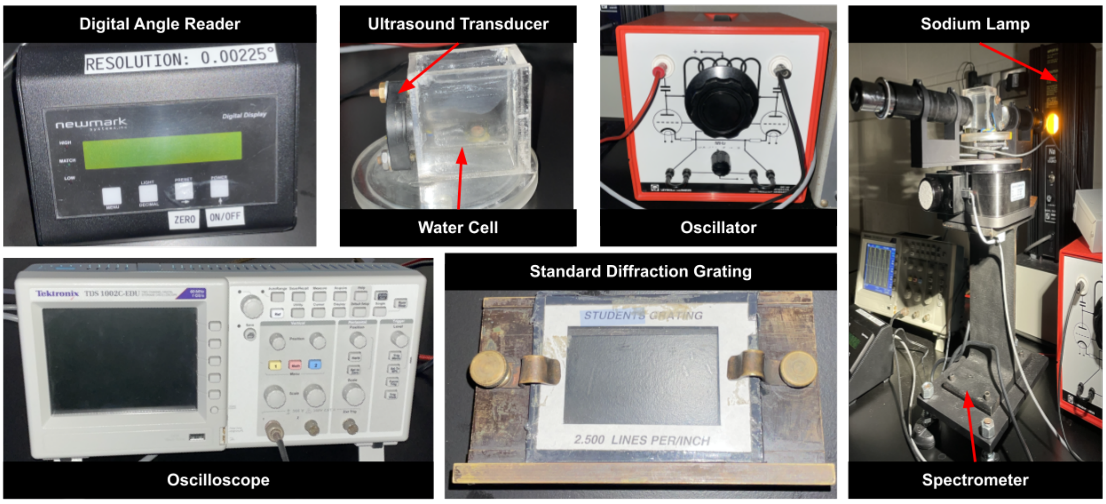

Figure 1: Labelled images of materials needed for the experiment (water and generator box excluded).

Equipment:

- Spectrometer
- Standard diffraction grating (2500 lines/inch)
- Sodium lamp
- Digital angle reader[1] (uncertainty ± 0.00125)
- Water cell
- Water (ideally distilled)
- Ultrasound transducer
- Oscilloscope[2] (uncertainty ± 0.000005)
- Oscillator
- Generator box

2.2 Methods

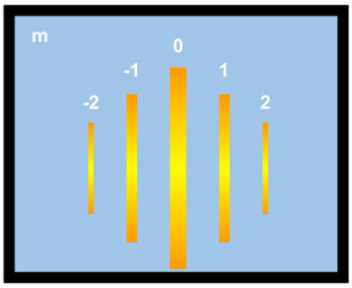

Figure 2: Diagram of the view of diffraction orders (m $= 0, \pm 1, \pm 2$) from the spectrometer.

2.2.1 Part 1: Standard Diffraction Grating

1. The sodium lamp was turned on and allowed to warm until it emitted a bright yellow light (about 10 minutes).
2. The instrument slit at the back was adjusted to obtain a clear view of the slit and cross-hairs.
3. The diffraction grating was placed on the rotating table in front of the spectroscope.

[1] The digital angle reader displays values in increments of 0.0025, so the uncertainty is calculated by dividing 0.0025 by 2, yielding 0.00125

[2] If the oscilloscope cannot measure frequency, a frequency counter will also be required for this experiment.

4. The lens was focused until two thin yellow lines, separated by a narrow gap, were seen, which occurs when the spectrometer is aligned with the sodium lamp.
5. The knob was slowly turned to rotate the table until the diffraction orders m = 0, m = ± 1, and m = ± 2 were visible. Refer to **Figure 2**.
6. The digital angle reader and the knob were used to rotate the table and measure the angles for the diffraction gratings specified in Step 5.
   (a) The cross-hairs were used to center each diffraction order for greater precision in angle readings.
   (b) At m = 0, the "Reset" button on the digital angle reader was clicked so $\Delta\theta = 0.00000$ when m = 0.

2.2.2 Part 2: Water-Filled Cell

1. Steps 1 and 3 from **Section 2.2.1** were repeated.
2. The cell was filled with water and placed on the rotating table in front of the spectroscope.
3. The ultrasound transducer was connected to the oscillator through the generator box. If using the frequency counter, it should be connected to the oscilloscope.
   (a) Do not connect the oscilloscope and oscillator directly, since its output voltage is near 300 V.
4. The ultrasonic beam was switched on and the frequency was adjusted to about 2.00 MHz.
5. Steps 5 and 6 from **Section 2.2.1** were repeated for several frequencies f_s between 1.80 and 2.10 MHz

3 Data and Analysis

Refer to **Appendix A** for uncertainty and error propagation and sample calculations.

Although the SI units for wavelength is meters, m, the measurements for all wavelengths in this experiment have been reported as either millimeters (mm), or nanometers, nm, for consistency. Additionally, λ is not computed for m = 0 as it is undefined due to $\Delta\theta$ being 0, which is indicated by a '*' in the table.

3.0.1 Part 1: Wavelength of Sodium Length

The aim for the first part of the experiment is determine the wavelength of the sodium lamp using the standard diffraction grating with a grating constant, N = 2500 lines/inch, by rearranging equation (1):

$$\lambda_L = \frac{dsin(\theta)}{m} \tag{4}$$

where m = diffraction order, λ_L = wavelength of the sodium lamp, d = slit spacing, and θ = angle of emergence. To determine the slit spacing, d, using the grating constant N:

$$d = \frac{1}{N} = \frac{1}{2500 \text{ lines/in}} = 0.0004 \text{ in/line} \approx 10160 \text{ nm} \tag{5}$$

Table 1, below, shows the readings of $\Delta\theta$ from the digital angle reader and the corresponding values of λ_L computed using equation (1) for each order of diffraction.

Table 1: Angular displacements for different diffraction orders for a standard diffraction grating

m	$\lvert\Delta\theta\rvert \pm 0.0075$ (°)	$\lambda_L \pm 0.00747$ (nm)
2	6.7925	600.83
1	3.4550	612.29
0	0.0000	*
-1	3.4425	610.01
-2	6.7950	601.21

The mean wavelength can be calculated to determine the wavelength of light from the sodium lamp:

$$\lambda_{L(avg)} = \frac{\lambda_{L(m=2)} + \lambda_{L(m=1)} + \lambda_{L(m=-1)} + \lambda_{L(m=-2)}}{4} = 606.08 \text{ nm} \pm 0.00747 \quad (6)$$

3.0.2 Part 2: Velocity of Ultrasonic Waves in Water and Bulk Modulus of Water

The aim for the second part of the experiment is to determine the velocity of ultrasonic waves in water and the bulk modulus of water.

Similar to (4), the following equation can be used to derive a relationship between the wavelength of the ultrasonic wave (λ_s) and the corresponding angle:

$$m'\lambda_L = d' sin(\theta') \quad (7)$$

where, m' = diffraction order, λ_L = average wavelength of the sodium lamp (determined in **3.1.1**), θ' = angle for the $m^{th'}$ order of diffraction, and d' = slit spacing. The diffraction spacing occurs such that:

$$d' = \lambda_s \quad (8)$$

Substituting equation (8) into into equation (7) and rearranging the equation in terms of λ_s:

$$\lambda_s = d' = \frac{m'\lambda_L}{sin(\theta')} \quad (9)$$

Table 2 [3] , shows the $\Delta\theta$ and λ_s for the water diffraction grating, for different frequencies from 1.80 MHz to 2.10 MHz. This range of frequencies was chosen to provide a basis for drawing a reliable trend.

Table 2: $\Delta\theta$ and λ_s for the Water Diffraction Grating at different frequencies (f_s)

(a) Frequency $f_s = 1.80121\ MHz$

m	$\lvert\Delta\theta\rvert' \pm 0.0075$ (°)	λ_s (nm)
2	0.07750	89615
1	0.04000	86815
0	0.00000	*
-1	0.04500	77169
-2	0.07750	89616

(b) Frequency $f_s = 1.90249\ MHz$

m	$\lvert\Delta\theta\rvert' \pm 0.0075$ (°)	λ_s (nm)
2	0.08250	84184
1	0.05250	66145
0	0.00000	*
-1	0.04500	77169
-2	0.08250	84184

(c) Frequency $f_s = 2.01801\ MHz$

m	$\lvert\Delta\theta\rvert' \pm 0.0075$ (°)	λ_s (nm)
2	0.10000	69452
1	0.05500	63138
0	0.00000	*
-1	0.04000	86815
-2	0.09250	75083

(d) Frequency $f_s = 2.10496\ MHz$

m	$\lvert\Delta\theta\rvert' \pm 0.0075$ (°)	λ_s (nm)
2	–	–
1	0.0550	63138
0	0.0000	*
-1	0.0450	77169
-2	–	–

The $\lambda_{s(avg)}$ for each f_s can be computed using the formula below, and is summarized in **Table 3**.

$$\lambda_{s(avg)} = \frac{\lambda_{s(m=2)} + \lambda_{s(m=1)} + \lambda_{s(m=-1)} + \lambda_{s(m=-2)}}{4} \quad (10)$$

Table 3: Frequency, f_s, vs. $\lambda_{s(avg)}$ for Ultrasonic Waves in Water

Frequency, f_s (MHz) ±0.000005	$\lambda_{s(avg)}$ (mm)	Uncertainty σ_{λ_s}
1.80121	0.85804	± 0.29263
1.90249	0.77921	± 0.23225
2.01801	0.73622	± 0.60073
2.10496	0.70154	± 0.30450

[3] The diffraction order $m = \pm 2$ were not visible, so $\lvert\Delta\theta\rvert$ is not reported, and hence, their λ_s values are not calculated, indicated by '–'.

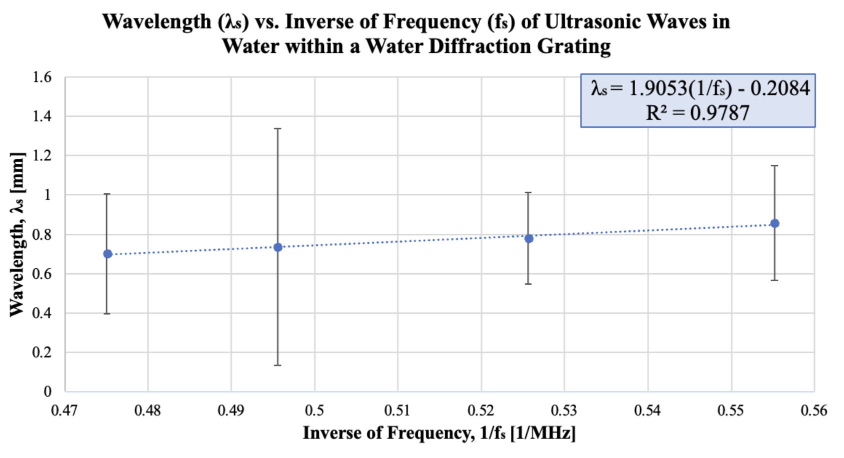

Figure 3: A scatter plot of $\frac{1}{f_s}$ vs. λ_s with a linear regression. The horizontal error bars for $\frac{1}{f_s}$ are barely visible, as the reading uncertainty relative to $\frac{1}{f_s}$ is quite small. The vertical error bars for λ_s are the reading uncertainty rather than the random, statistical uncertainty.

The wavelength was plotted against the inverse of the frequency, producing **Figure 3**, to determine the velocity of the ultrasonic wave, v_s. The fit was composed of a slope of 1.9053mm, and an intercept of $-0.2084\frac{\text{mm}}{\text{MHz}}$. As v_s is approximately equal to the slope of the fit:

$$v_s = 1.9053\ \frac{\text{mm}}{\text{MHz}} = 1905.3\ \frac{m}{s} \pm 45.6\ \frac{m}{s} \tag{11}$$

Rearranging (3) to determine B, and letting the literature value for water density as $1000\frac{\text{kg}}{\text{m}^3}$ [2] yields:

$$B = v_s^2 \rho = 1905.3^2\ (\frac{m^2}{s^2}) \times 1000\ (\frac{kg}{m^3}) = 3.63 \pm 0.17\ \text{GPa} \tag{12}$$

4 Discussion and Conclusion

In this experiment, the wavelength of light produced by a sodium lamp, λ_L, was calculated to be λ_L was found to be 606.08 ± 0.00747nm. This value was used to determine the velocity of sound, v_s, and the bulk modulus of water, B, by measuring the relationship between diffraction orders, m, the difference in angle $\Delta\theta$, frequency, f_s, and wavelength λ_s and applying equations (1), (2), and (3).

The following equation was used to propagate errors:

$$\%_{Error} = \frac{Literature\ Value - Experimental\ Value}{Experimental\ Value} \times 100| \tag{13}$$

The experimental value of v_s was found to be $1.905.3 \pm 45.6\frac{\text{m}}{\text{s}}$, which agrees within 21.4% of the literature value of $1497.5 \pm 0.5\frac{\text{m}}{\text{s}}$ [3], as per equation (13), and although not heavily accurate, is precise as per the goodness of fit criteria evaluated later in this section. The experimental value of B was found to be 3.63 ± 0.17GPa, which agrees within 41.7% of the literature value of 1.96 GPa [4], as per equation (13), which is also not quite accurate. Reasons for the limited accuracy between the experimental and literature values will be discussed later in this section.

The coefficient of determination, R^2 of the fit from **Figure 3** was 0.9787, and as that is close to 1, indicated a good fit of data.

Figure 4 displays a residual plot, and from it, it was determined that residuals range from about -0.015 to 0.01. This range is a marginal difference from the ideal value of 0, which reaffirmed the fit resulting from **Figure 3** was a good fit.

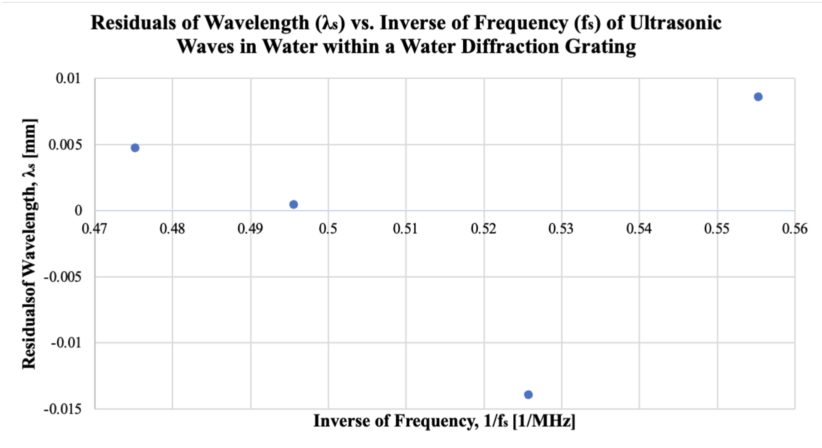

Figure 4: A scatter plot of $\frac{1}{f_s}$ vs. the residuals of λ_s relative to the fit yielded in **Figure 3**.

The chi-squared value, χ^2, of this fit was calculated to be 0.00156. This is much closer to 0 than the ideal value of 1, which implied an overfit. However, since the fit is linear (a polynomial of degree 1), it was not logical to assume that it is an overfit since choosing a fit with a lower degree would be a constant value. Hence, the fit $\lambda_s = 1905.3 \times \frac{1}{f_s} - 0.2084$ was a good fit, confirming the results of **Figure 4**.

The results of R^2, the residual plot, and χ^2 illustrate that the results agreed with a linear dependence of d, which equals λ_s, against $\frac{1}{f_s}$, proving that sound waves do not disperse in water.

Two main sources of error potentially affected the results of this experiment, which explain the large uncertainties in values such as λ_s. Firstly, measured values of θ for diffraction orders were less than 0.01°, and parallax error, especially dependent on the height and angle from which the diffraction order was viewed through the spectrometer, meant that even marginal differences in the value of θ could have significant impacts on final results. Additionally, as the frequency increased, the spacing between and resolution of diffraction orders decreased. Given the limits of available equipment and human vision, it was more difficult to differentiate between diffraction orders. For certain frequencies and diffraction orders, data could not be confidently collected, reducing the robustness of the dataset to use in calculations. The effects of both of these errors could be mitigated by conducting several trials of the same setup to accommodate the variance in θ caused by the parallax effect and amend the lack of sufficient data caused by the restricted capabilities of human vision and experiment equipment.

In conclusion, the wavelength of the sodium lamp was found to be 606.08 ± 0.00747nm, the velocity of sound was found to be $1.905.3 \pm 45.6 \frac{\mathrm{m}}{\mathrm{s}}$, which differs from literature values by 21.4% [3] and the bulk modulus of water was found to be 3.63 ± 0.17GPa, which deviates by 41.7% of the literature value [4]. Thus, while this experiment was semi-successful in analyzing an ultrasonic standing wave in water as an optical diffraction grating to determine the velocity of ultrasonic waves in water and the bulk modulus of water for adiabatic compression, it could have been more successful if the impact of the sources of error from parallax and human vision and equipment resolution restrictions were reduced.

5 References

[1] C. Lee and R. Serbanescu, "The velocity of Ultrasonic Waves in Water by the Debye-Sears Effect", https://q.utoronto.ca/courses/324674/files/27571494?module_item_id=5066355 (accessed Oct. 31, 2023).

[2] "Density of water, distilled in 285 units and reference information", aqua-calc.com, https://www.aqua-calc.com/page/density-table/substance/water-coma-and-blank-distilled (accessed Oct. 31, 2023).

[3] K. Kamide, "Molecular Properties of Cellulose and Cellulose Derivatives", sciencedirect.com, https://www.sciencedirect.com/science/article/abs/pii/B9780444822543500059 (accessed Oct. 31, 2023).

[4] J. Gouvea, K. Nordstrom, and J. Redish, “Bulk modulus – liquids”, compadre.org, `https://www.compadre.org/nexusph/course/Bulk_modulus_--_liquids#:~:text=The%20bulk%20modulus%20for%20water,9%20Pa%20for%20stainless%20steel` (accessed Oct. 31, 2023).

6 Appendix A: Uncertainty Propagation

For the uncertainty in $\Delta\theta$, there are two sources: the precision of the digital angle reader (± 0.00125) and the parallax error in measured angle.
To quantify the uncertainty of the parallax effect, the "width" of the diffraction order m = 0, since it is the thickest diffraction order and therefore has the largest uncertainty, was measured in terms of $\Delta\theta$ and then divided by 2, where θ_1 and $\theta 2$ are the left- and right-most measurements of the diffraction order.

$$\text{Uncertainty,}\Delta\theta = \frac{\theta_1 - \theta_2}{2} \tag{14}$$

The digital angle reader was not tared for these measurements since only the difference in angle was being measured. Additionally, this data was collected after the experiment was conducted instead of during it, so the frequencies are not the same, although very similar. However, this does not significantly affect the uncertainty propagation. To determine the major source of uncertainty in the calculations, both the instrumental uncertainties from the digital angle reader and experimental uncertainty from the parallax error need to be compared.

6.1 Part 1: Wavelength of Sodium Length Uncertainty Propagation

To propagate experimental uncertainties due to the parallax effect, θ_1 and θ_2 can be computed by finding the absolute difference in the width of the diffraction orders, m = 0, for the standard diffraction grating:

$$\theta_1 = 0.0025^\circ \tag{15}$$
$$\theta_2 = -0.0125^\circ \tag{16}$$

Substituting θ_1 and θ_2 into equation (18) to find the experimental uncertainty due to parallax error:

$$\text{Uncertainty,}\Delta\theta = \frac{\theta_1 - \theta_2}{2} = \frac{(0.0025^\circ) - (-0.0125^\circ)}{2} = \frac{0.015^\circ}{2} = 0.0075^\circ \tag{17}$$

Since the experimental uncertainty is significantly greater than the instrumental uncertainty ($0.0075 > 0.0005$), the uncertainty in the angle measurements is considered to be $\pm$ 0.0075.

In order to propagate the uncertainty in λ_L the percentage uncertainties in d, m and $Sin(\theta)$ need to be summed. Since d and m are constants, they have no uncertainties. As such the uncertainty in λ_L is only contributed by the uncertainty in $sin(\theta)$. To compute the uncertainty of a sine function, the general formula for the uncertainty of a function of one variable can be utilized:

$$\sigma_{f(x)} = \left|\frac{df}{dx}\right| \sigma_x \tag{18}$$

Since λ_L has a sine function, the uncertainty in λ_L is given by uncertainty in $sin(\theta)$

$$\sigma_{sin(\theta)} = \left|cos(\theta)\right| \sigma_x \tag{19}$$

Table 4: Propagating the Average Uncertainty in λ_L

m	$\Delta\theta(^\circ)$	$\lvert cos(\Delta\theta)\rvert$	σ_x	$\sigma_{sin(\theta)}$
2	-6.7925	0.99298	0.00750	0.00745
1	-3.4550	0.99818	0.00750	0.00749
-1	3.4425	0.99819	0.00750	0.00749
-2	6.7950	0.99297	0.00750	0.00745
Average Uncertainty in λ_L*				0.00747

*This is the average uncertainty in the average wavelength ($\lambda_{L(avg)}$) of the sodium lamp. We will assume this as the major source uncertainty later in calculations of part 2.

6.2 Part 2: Velocity of Ultrasonic Waves in Water and Bulk Modulus of Water Uncertainty Propagation

Table 5: Frequency, f_s (MHz) vs. Uncertainty from Parallax Error of $\Delta\theta$

Frequency, f_s (MHz)	θ_1 (°)	θ_2 (°)	Uncertainty for $\Delta\theta$ (°)
1.80190	0.0025	-0.0125	0.0075
1.90120	0.0025	-0.0125	0.0075
2.01132	0.0000	-0.0150	0.0075
2.10715	0.0000	-0.0150	0.0075

Since the uncertainty of the parallax effect is greater than that of the digital angle reader, the uncertainty of ± 0.0075 is used in calculations. To determine the uncertainties in λ_s, the individual uncertainties need to be computed for each frequency for each order. However, for each frequency and order of diffraction, the value of $\left|cos(\Delta\theta)\right|$ term (23) approximates to 1.00000 within the degree of instrumental precision of 5 decimal places. As such, the term $\sigma_{sin(\theta)} = \sigma_x = 0.00750$ for all frequencies and orders of diffraction.

In λ_s, the sources of uncertainties are the uncertainty in the sine function $\sigma_{sin(\theta)}$ and λ_L, and both must be accounted for by adding the fractional uncertainties of both.

Fractional uncertainty in λ_L is given by:

$$\sigma_{\lambda_L} = \frac{0.00747}{606.08} \times 100 = 0.0000123 \approx 0.00001\ \% \tag{20}$$

Fractional uncertainty in the sine function where Δx are the individual uncertainties and x are the actual values of $sin(\theta)$ for each frequency:

$$\sigma_{f(\theta)=sin(\theta)} = \Sigma\frac{\Delta x}{x} \tag{21}$$

Therefore the overall uncertainty in λ_s for each frequency is given by:

$$\sigma_{\lambda_s} = \sigma_{\lambda_L} + \sigma_{sin(\theta)} \tag{22}$$

The table below shows the summarized values for the propagated uncertainties for each frequency.

Table 6: Frequency, f_s (MHz) vs. Uncertainty of σ_{λ_s} (in mm)

Frequency, f_s (MHz)	σ_{λ_L}	$\sigma_{sin(\theta)}$	σ_{λ_s}
1.80190	0.00747	34.104 %	0.29263
1.90120	0.00747	29.841 %	0.23252
2.01132	0.00747	81.957 %	0.60073
2.10715	0.00747	43.405 %	0.30450

Based on the calculations, the variance of $y(x)$ led to the larger source of uncertainty. As such the variance was used to propagate the uncertainty in the velocity of water (here N = 4, i.e. the number of x-y pairs):

$$s^2_{y,x} = \frac{1}{N-2}\Sigma(y_i - [b + m_i x])^2 = 7.66538 \tag{23}$$

$$\Delta = N\Sigma x_i^2 - (\Sigma x_i)^2 = 0.01473 \tag{24}$$

Since the velocity was computed using the slope of linear fit, only the uncertainty in m is considered:

$$s_m = \sqrt{N\frac{s^2_{y,x}}{\Delta}} = 45.62457 \tag{25}$$

For computing the uncertainty in bulk modulus the percentage uncertainty in v_s was added twice to yield an absolute uncertainty in B of ± 0.17 GPa.